METU - NCC

LINEAR ALGEBRA MIDTERM EXAM 1	
Code : *MAT 260* Acad.Year: *2014-2015* Semester : *FALL* Date : *01.11.2014* Time : *09:40* Duration : *90 min*	Last Name: Name : KEY Student # : Signature : 5 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1. (10)	2. (30)	3. (25)	4. (25)	5. (10)			

1. *(10pts)* Let $U = \{(x, y, z) : 2x + y - 3z = 0\} \subseteq \mathbb{R}^3$. Find a basis for U. Justify your answer, that is, they are linearly independent and span the subspace U.

$\vec{a}_1 = (1, -2, 0)$, $\vec{a}_2 = (0, 3, 1)$ are linearly independent vectors from $\mathcal{U}$.

For every $\vec{u} = (x, y, z) \in \mathcal{U}$ we have

$$\vec{u} = x(1, -2, 0) + z(0, 3, 1) = x\vec{a}_1 + z\vec{a}_2 \in \mathrm{Span}\{\vec{a}_1, \vec{a}_2\}.$$

Thus $\mathcal{U} = \mathrm{Span}\{\vec{a}_1, \vec{a}_2\} \Rightarrow (\vec{a}_1, \vec{a}_2)$ is a basis for $\mathcal{U}$.

2. *(15+15pts)* Determine if the following sets are linearly independent. Justify your answers.

(a) $E = \{\underset{\vec{u}_1}{\underset{\shortparallel}{(1,2,3,-1)}}, \underset{\vec{u}_2}{\underset{\shortparallel}{(2,1,-1,1)}}, \underset{\vec{u}_3}{\underset{\shortparallel}{(1,-4,-11,5)}}\} \subseteq \mathbb{R}^4$.

Consider a linear relation $\lambda \vec{u}_1 + \mu \vec{u}_2 + \theta \vec{u}_3 = \vec{0}$, that is,

$$\begin{cases} \lambda + 2\mu + \theta = 0 & \longrightarrow \quad 3\mu + 6\theta = 0 \\ 2\lambda + \mu - 4\theta = 0 & \longrightarrow \quad 5\lambda - 15\theta = 0 \\ 3\lambda - \mu - 11\theta = 0 & \\ -\lambda + \mu + 5\theta = 0 & \end{cases}$$

It follows that $\mu + 2\theta = 0$, $\lambda = 3\theta$. Put $\theta = 1$. Then $\lambda = 3$, $\mu = -2 \Rightarrow 3\vec{u}_1 - 2\vec{u}_2 = -\vec{u}_3$, that is, E is a linearly independent subset.

(b) $E = \{2\chi_a + \chi_b - \chi_c, \quad \chi_a - 2\chi_b + \chi_c\} \subseteq \text{Fun}(S)$, where $S = \{a, b, c\}$.

Being $\mathbb{R}^3$ an isomorphic copy of Fun(S), we have $\lambda(2,1,-1) + \mu(1,-2,1) = 0 \Rightarrow$

$$\Rightarrow \begin{cases} 2\lambda + \mu = 0 \\ \lambda - 2\mu = 0 \\ -\lambda + \mu = 0 \end{cases} \Rightarrow \lambda = \mu = 0,$$

that is, E is a linearly independent subset.

3. *(25pts)* Consider the subset $E = \{\overset{\vec{u}_1}{(1,-1,2,1)}, \overset{\vec{u}_2}{(-1,2,0,1)}, \overset{\vec{u}_3}{(-2,2,-4,-2)}\} \subseteq \mathbb{R}^4$ of vectors. Find the subspace $Span(E)$ spanned by E in terms of the linear equations in $\mathbb{R}^4$.

First note that $\vec{u}_3 = -2\vec{u}_1$, therefore Span(E) $= \text{Span}\{\vec{u}_1, \vec{u}_2\}$. But $\lambda\vec{u}_1 + \mu\vec{u}_2 =$

$$= (\underset{x}{\underset{\parallel}{\lambda-\mu}}, \underset{y}{\underset{\parallel}{-\lambda+2\mu}}, \underset{z}{\underset{\parallel}{2\lambda}}, \underset{w}{\underset{\parallel}{\lambda+\mu}}), \text{ that is,}$$

$$\begin{cases} \lambda - \mu = x & \longrightarrow \mu = y + x \;\cancel{+\lambda} \\ -\lambda + 2\mu = y & \longrightarrow -z/2 + 2y + 2x = y \\ 2\lambda = z & \longrightarrow \lambda = \dfrac{z}{2} \\ \lambda + \mu = w & \longrightarrow z/2 + y + x = w \longrightarrow z + 2x + 2y = 2w \end{cases}$$

Thereby Span$(E) \subseteq \{z = 2y + 4x,\ z + 2x + 2y = 2w\}$

But the solution set of the system

$$\begin{cases} 4x + 2y - z = 0 \\ 2x + 2y + z - 2w = 0 \end{cases} \quad \text{is } \{(a, b, 4a+2b, 3a+2b)\}$$

Whence Span$(E) = \{z = 2y + 4x,\ z + 2x + 2y = 2w\}$.

4. *(25pts)* Consider the set $E = \{x-1, (x+1)(x-2)\}$ of vectors from the space $\mathcal{P}_3(\mathbb{R})$. Extend the set E up to a basis of $\mathcal{P}_3(\mathbb{R})$. Justify your answer.

Note that $\lambda(x-1) + \mu(x^2 - x - 2) = -\lambda + \lambda x - 2\mu - \mu x + \mu x^2$

$$= \underbrace{(-\lambda - 2\mu)}_{= a} + \underbrace{(\lambda - \mu)}_{= b} x + \underbrace{\mu}_{= c} x^2$$

$$\begin{cases} -\lambda - 2\mu = a \\ \lambda - \mu = b \longrightarrow \lambda = b + c \\ \mu = c \end{cases} \longrightarrow b + c + 2c = -a \longrightarrow a + b + 3c = 0$$

Hence $\mathrm{Span}(E) = \{a + bx + cx^2 : a + b + 3c = 0\}$.

In particular, $1 \notin \mathrm{Span}(E)$, that is $\{x-1, (x+1)(x-2), 1\}$ is a lin. independent subset. Further, $\mathrm{Span}\{x-1, (x+1)(x-2), 1\} \subseteq \mathcal{P}_2(\mathbb{R})$, therefore

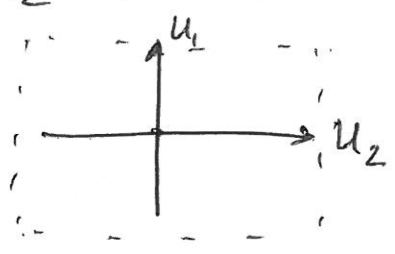
$\{x-1, (x+1)(x-2), 1, x^3\}$ is a basis for $\mathcal{P}_3(\mathbb{R})$.

5. *(10pts)* Let U_1, U_2 be two subspaces of a vector space V. Show that $U_1 \cap U_2$ is a subspace of V. What about the union $U_1 \cup U_2$, is it a subspace yet?

If $x, y \in U_1 \cap U_2$ then $\lambda x + \mu y \in U_1 \cap U_2$, for both U_1, U_2 are subspaces.

The union $U_1 \cup U_2$ is not a subspace in general.

For example,



in $\mathbb{R}^2$.